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A NOTE ON MATRICES WITH ZERO TRACE

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1. It is known that an $n\times n$ matrix A with elements in a field F can be written as a commutator $XY-YX$ over F, if and only if the trace of A is zero. This result was proved by Shoda [1] in the case that the field has characteristic zero and was extended by Albert and Muckenhoupt [2] to fields of arbitrary characteristic. In this note we consider Shoda's result when F is the real or complex numbers and we also derive some results when A is hermitian or skew-hermitian.

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2. We shall make use of the following two results of W. V. Parker [3].

LEMMA 1. *If A is an $n\times n$ matrix of complex numbers and trace A is zero, then there exists a unitary matrix U so that UAU^* has all its main diagonal elements equal to zero.*

LEMMA 2. *If A is a real $n\times n$ matrix with trace zero, then there exists a real orthogonal matrix T so that TAT^t has zero main diagonal.*

Our main result is the following:

THEOREM 1. *If A is an $n\times n$ complex matrix and trace A is zero, then there exist matrices X and Y so that $A=XY-YX$, where X is hermitian and Y has trace zero.*

Proof. By Lemma 1 we can find a unitary matrix U so that $UAU^*=B=(b_{ij})$ has zero diagonal. Let $D=\operatorname{diag}(d_1, d_2, \cdots, d_n)$ where the d_i are real and distinct, and let $Y_1=(y_{ij})$ where $y_{ij}=b_{ij}/(d_i-d_j)$ when $i\neq j$ and $y_{ii}=0$, $i, j=1, 2, \cdots, n$. Then $B=DY_1-Y_1D$ and thus $A=XY-YX$ where $X=U^*DU$, $Y=U^*Y_1U$, and we see that $X^*=X$ and trace $Y=0$.

COROLLARY 1. *If, in addition, A is hermitian, then it can be written as $XY-YX$, where X is hermitian and Y is skew-hermitian.*

COROLLARY 2. *If A is skew-hermitian with trace zero, it can be written as $XY-YX$ where both X and Y are hermitian.*

REMARK. If A is a real matrix with trace zero then Theorem 1 and its corollaries hold if we replace "hermitian" and "skew-hermitian" by "symmetric" and "skew-symmetric," respectively. Lemma 2 is used to prove these facts.

If, in Corollary 1, we replace X by $B=(1/\sqrt{2})(X-Y)$ and Y by $B^*=(1/\sqrt{2})(X+Y)$, we get the following theorem due to R. C. Thompson [4].

THEOREM 2. *If A is a hermitian matrix and trace $A=0$, then A can be written as BB^*-B^*B.*

It is also true that Corollary 1 follows from Thompson's result on replacing B by $(1/\sqrt{2})(X-Y)$ where $X=(1/\sqrt{2})(B^*+B)$ and $Y=(1/\sqrt{2})(B^*-B)$.

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SOLUTIONS OF $x^4+y^4=z^4$ IN 2×2 INTEGRAL MATRICES

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A. Aigner [1] investigated solutions of $x^4+y^4=z^4$ in quadratic domains. In this note, solutions are found in the ring Γ of 2×2 matrices with integer elements. Let 0 and I be the zero and identity matrices respectively.

THEOREM. *There exist solutions of $A^4+B^4=C^4$, where A, B and C are in Γ and $A^4\neq 0$, $B^4\neq 0$, $C^4\neq 0$.*

Proof. Set

$$A=\begin{bmatrix}0 & a\\ b & 0\end{bmatrix},\qquad B=\begin{bmatrix}0 & c\\ d & 0\end{bmatrix},\qquad C=\begin{bmatrix}0 & e\\ f & 0\end{bmatrix}.$$

Then $A^4=(ab)^2I$, and there are similar expressions for B^4 and C^4. Thus, $A^4+B^4=C^4$ if and only if $(ab)^2+(cd)^2=(ef)^2$. But using the well-known solution to the Diophantine equation $x^2+y^2=z^2$, we can set $b=d=f=1$, and $a=2mn$, $c=m^2-n^2$, $e=m^2+n^2$, obtaining

$$\begin{bmatrix}0 & 2mn\\ 1 & 0\end{bmatrix}^4+\begin{bmatrix}0 & m^2-n^2\\ 1 & 0\end{bmatrix}^4=\begin{bmatrix}0 & m^2+n^2\\ 1 & 0\end{bmatrix}^4.$$

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